Thus the best circulator configuration for broadest bandwidth (due to minimum phase displacement of the modes) and lowest loss was the single-sided unit having a ferrite rod length of $3\lambda/4$. Because of the extremely nonlinear dependence of the propagation constants on the normalized ferrite diameter at 60 GHz ($m_s = 0.24$) there are two design options with widely different geometries and bias magnetic fields. The option having the longer and slimmer ferrite rod required significantly less bias field due to a smaller demagnetizing factor.

IV. CONCLUSIONS

A simple design procedure for the widely used partial height ferrite waveguide circulator has been formulated. It eliminates the need for sophisticated computer programs and/or elaborate experimental design techniques. Excellent agreement between theory and experiment has been obtained with circulators operating in the "turnstile" mode. Even greater isolation bandwidths can be achieved by stagger tuning with the standard disk resonator mode $TM_{\pm 1,1,0}$ or the $TM_{0,1,\delta}$ resonance induced by a pin inserted along the axis of the ferrite rod. However, since design formulas for these configurations are much more difficult to obtain, experimental design techniques such as using an eigenvalue measuring set [4] must be used.

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Letters

Guided Waves Along Graded Index Dielectric Rod

RYOZO YAMADA AND YASUNOBU INABE

Abstract—**By a modification of the Kurtz and Streifer procedure, the coupled second-order differential equations for the field components of guided modes along a graded index dielectric rod surrounded by a homogeneous medium were solved directly. Using the results, the eigenvalue equations which are consistent with those of the simple core-cladding-type dielectric fiber in the region near the cutoffs were obtained.**

Optical transmission through a graded index glass fiber has been given extensive attention. The theoretical study of guided waves in a focusing medium has been done by many investigators. Among them, Kurtz and Streifer have treated this problem on the basis of the circular cylindrical coordinates and solved approximately the linear homogeneous fourth-order differential equation, and applied the results to the wave propagation guided by an enclosed circular cylindrical graded index dielectric rod [1]–[3].

In this letter, we deal with the guided waves along a graded index dielectric rod such as Selfoc. By a modification of the Kurtz and Streifer procedure, we solve directly the coupled second-order differential equations approximately and apply the results to the guided waves along the rod. We adopt here the notations that have been defined in their paper [1].

We assume that the dielectric constant distribution is in the form

$$\epsilon = \epsilon_1\left(1 - \delta\left(\frac{r}{a}\right)^2\right), \qquad r < a \tag{1}$$

and

$$\epsilon = \epsilon_2, \qquad r > a \tag{2}$$

where

$$\epsilon_1 > \epsilon_2.$$

We express the axial components of electric and magnetic fields as

$$E_\zeta = E_{n\zeta} \exp[j(\omega t - \beta z)] \cos(n\phi + \theta) \tag{3}$$

$$H_\zeta = H_{n\zeta} \exp[j(\omega t - \beta z)] \sin(n\phi + \theta) \tag{4}$$

and further we write

$$E_{n\zeta} = [\epsilon_1(1-\chi)]^{-1/4}\phi \quad \text{and} \quad H_{n\zeta} = [\epsilon_1(1-\chi)]^{1/4}\psi/\eta_0 \tag{5}$$

where

$$\chi = 1 - \beta^2/k^2\epsilon_1 \qquad \eta_0 = (\mu_0/\epsilon_0)^{1/2}$$

instead of

$$\phi = \epsilon_1^{1/4}E_\zeta \qquad \psi = -i\epsilon_1^{-1/4}\eta_0 H_\zeta$$

in the paper [1].

We obtain the wave equations for ϕ and ψ in the rod:

$$\frac{d^2\phi}{dz^2} + \left[\frac{1}{z} + \frac{2z}{1-z^2} - 2\chi z\right]\frac{d\phi}{dz} + \left[b^2(1-z^2) - \frac{n^2}{z^2}\right]\phi = \frac{-2n\psi}{1-z^2} + 2n\chi\psi \tag{6}$$

$$\frac{d^2\psi}{dz^2} + \left[\frac{1}{z} + \frac{2z}{1-z^2}\right]\frac{d\psi}{dz} + \left[b^2(1-z^2) - \frac{n^2}{z^2}\right]\psi = \frac{-2n\phi}{1-z^2} \tag{7}$$

where

$$z = (\delta/\chi)^{1/2}(r/a) \qquad b^2 = (ka\chi)^2\epsilon_1/\delta.$$

Since χ is small for the guided modes whose fields are bounded near the region of maximum permittivity, we neglect the terms $2\chi z(d\phi/dz)$, $2n\chi\psi$, and reduce (6) and (7) to

$$\frac{d^2\phi}{dz^2} + \left[\frac{1}{z} + \frac{2z}{1-z^2}\right]\frac{d\phi}{dz} + \left[b^2(1-z^2) - \frac{n^2}{z^2}\right]\phi + \frac{2n\psi}{1-z^2} = 0 \tag{8}$$

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The authors are with the Department of Electronic Engineering, Shizuoka University, Jyohoku, Hamamatsu, Japan.

$$\frac{d^2\psi}{dz^2} + \left[\frac{1}{z} + \frac{2z}{1-z^2}\right]\frac{d\psi}{dz} + \left[b^2(1-z^2) - \frac{n^2}{z^2}\right]\psi + \frac{2n\phi}{1-z^2} = 0. \quad (9)$$

We add (8) and (9):

$$\frac{d^2}{dz^2}(\phi+\psi) + \left[\frac{1}{z} + \frac{2z}{1-z^2}\right]\frac{d}{dz}(\phi+\psi) + \left[b^2(1-z^2) - \frac{n^2}{z^2} + \frac{2n}{1-z^2}\right](\phi+\psi) = 0. \quad (10)$$

We substract (9) from (8):

$$\frac{d^2}{dz^2}(\phi-\psi) + \left[\frac{1}{z} + \frac{2z}{1-z^2}\right]\frac{d}{dz}(\phi-\psi) + \left[b^2(1-z^2) - \frac{n^2}{z^2} - \frac{2n}{1-z^2}\right](\phi-\psi) = 0. \quad (11)$$

If we write the solutions of (10) and (11) that are bounded at $z = 0$ as G_1 and G_2, respectively, we can express ϕ and ψ as

$$\psi = \Sigma A_jG_j \qquad \phi = \Sigma (-1)^{j+1}A_jG_j, \qquad j = 1,2. \quad (12)$$

Using the above expressions of E_ζ, H_ζ and Maxwell's equations, the transverse components of the fields can be written as

$$E_r = -j\Gamma(1-\chi)^{1/4}\cos(n\phi+\theta)\ \Sigma(-1)^{j+1}A_j\Phi_j \quad (13)$$

$$E_\phi = j\Gamma(1-\chi)^{1/4}\sin(n\phi+\theta)\ \Sigma A_j\Phi_j \quad (14)$$

$$H_r = -j\frac{\Gamma\epsilon_1}{\eta_0(1-\chi)^{1/4}}\left[(1-\chi)\ \Sigma A_j\Phi_j + \frac{n\chi}{z}\ \Sigma A_jG_j\right] \cdot \sin(n\phi+\theta) \quad (15)$$

$$H_\phi = -j\frac{\Gamma\epsilon_1}{\eta_0(1-\chi)^{1/4}}\left[(1-\chi z^2)\ \Sigma(-1)^{j+1}A_j\Phi_j - \frac{n\chi}{z}\ \Sigma A_jG_j\right] \cdot \cos(n\phi+\theta) \quad (16)$$

where

$$\Gamma = \delta^{1/2}/(ka\epsilon_1^{3/4}\chi^{3/2}) \qquad \Phi_j = \frac{z(dG_j/dz) \pm nG_j}{z(1-z^2)} \quad (17)$$

and Φ_j are solutions of the following equations which are bounded at $z = 0$:

$$\frac{d^2\Phi_j}{dz^2} + \frac{1}{z}\frac{d\Phi_j}{dz} + \left[b^2(1-z^2) - \frac{(n\mp 1)^2}{z^2}\right]\Phi_j = 0, \qquad j = 1,2 \quad (18)$$

and

$$G_j = -\frac{1}{b^2}\left[\frac{d\Phi_j}{dz} \mp \frac{(n\mp 1)}{z}\Phi_j\right]. \quad (19)$$

The solutions Φ_j are expressed in terms of Whittaker's functions as [6]

$$\Phi_j = z^{-1}M_{b/4,(n\mp 1)/2}(bz^2). \quad (20)$$

The axial components of the guided modes in the outer medium are [4]

$$E_\zeta = B\epsilon_1^{-1/4}(1-\chi)^{-1/4}K_n(\lambda r)\exp[j(\omega t - \beta z)]\cos(n\phi+\theta) \quad (21)$$

$$H_\zeta = C\frac{\epsilon_1^{1/4}(1-\chi)^{1/4}}{\eta_0}K_n(\lambda r)\exp[j(\omega t - \beta z)]\sin(n\phi+\theta) \quad (22)$$

where $K_n(\lambda r)$ is the modified Bessel functions of the second kind:

$$\lambda^2 = \beta^2 - k^2\epsilon_2. \quad (23)$$

The transverse components of the fields in the cladding can be expressed in terms of these E_ζ, H_ζ [4], [5].

Subjecting these fields to the boundary conditions provides the eigenvalue equation

$$-\frac{\eta + \eta_1 + n/w^2}{\eta + \eta_2 - n/w^2} = \frac{\eta + (1+\Delta)\eta_1 + n/w^2}{\eta + (1+\Delta)\eta_2 - n/w^2} \quad (24)$$

where

$$\eta = \frac{K_n'(w)}{wK_n(w)}, \quad \eta_1 = \frac{\Phi_1(u)}{b^{3/2}u^{1/2}G_1(u)}, \quad \eta_2 = \frac{\Phi_2(u)}{b^{3/2}u^{1/2}G_2(u)}$$

$$w = \lambda a, \quad u = bz^2|_{r=a} = ka(\epsilon_1\delta)^{1/2}, \quad \Delta = [\epsilon_1(1-\delta) - \epsilon_2]/\epsilon_2.$$

For $n = 0$, the two solutions of (18) coincide, $\Phi_1 = \Phi_2$, and (24) yields two eigenvalue equations for the axial symmetric TE and TM modes:

$$\eta + \eta_1 = 0 \ \text{ TE modes} \qquad \eta + (1+\Delta)\eta_1 = 0 \ \text{ TM modes.} \quad (25)$$

For $n \geq 1$. In the case where the difference between ϵ_1 and ϵ_2 is small, we can write the two sets of roots of (24), which is a quadratic equation in η, as

$$\eta + \frac{n}{w^2} = -\frac{2+\Delta}{2}\eta_1 + 0(\Delta^2) \qquad \eta - \frac{n}{w^2} = -\frac{2+\Delta}{2}\eta_2 + 0(\Delta^2). \quad (26)$$

To verify the validity of these equations, we examine these equations in the region near the cutoffs of the various modes by taking $w \to 0$. In the case where ϵ_1 and ϵ_2 have a finite difference and δ/a^2 is small, b becomes large as $w \to 0$ ($\beta^2 \to k^2\epsilon_2$). In this case we use the asymptotic expansions of Whittaker's functions [6], and express Φ_1, Φ_2 in terms of the Bessel functions as

$$\Phi_1 \sim J_{n-1}(bz) \qquad \Phi_2 \sim J_{n+1}(bz). \quad (27)$$

Then the eigenvalue equations (25) for $n = 0$ become

$$\frac{K_1(w)}{wK_0(w)} = -\frac{J_1(v)}{vJ_0(v)} \qquad \frac{K_1(w)}{wK_0(w)} = -(1+\Delta)\frac{J_1(v)}{vJ_0(v)} \quad (28)$$

where

$$v = bz|_{r=a} = ka(\epsilon_1\chi)^{1/2}$$

and the equations (26) for $n \geq 1$ become

$$\frac{K_{n\mp 1}(w)}{wK_n(w)} = \pm\frac{2+\Delta}{2}\frac{J_{n\mp 1}(v)}{vJ_n(v)}. \quad (29)$$

These eigenvalue equations in the region near the cutoffs are consistent with those of the simple core-cladding-type dielectric rod [4], [5].

In the case of Selfoc, the value of u is of the order of several hundreds, and the eigenvalues calculated in the region far from the cutoffs lie near the values that satisfy the equation

$$b = 4m + 2n, \qquad m = 1,2,\cdots \quad (30)$$

and the eigenfunctions Φ_j are

$$\Phi_j \doteqdot z^{n\mp 1}\exp(-bz^2/2)L_{m-1/2\pm 1/2}^{n\mp 1}(bz^2) \quad (31)$$

where L is the Laguerre polynomial.

In conclusion, we solved directly the coupled second-order differential equations by a modification of the Kurtz and Streifer procedure, and we may evaluate the effect of neglecting the terms $2\chi z(d\phi/dz), 2n\chi\psi$ in (6) by a perturbation method. Using the results, we obtained the eigenvalue equations which are consistent with those of the simple core-cladding-type dielectric rod in the region near the cutoffs. Based upon the results of this study, we plan to investigate the optical transmission through the graded index glass fibers.

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